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A SURVEY OF RENEWAL THEORY WITH EMPHASIS ON APPROXIMATIONS, BOUNDS, AND APPLICATIONS

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The models of Renewal Theory are reviewed, with particular emphasis on their practical applications. Bounds and approximations to various quantities are given, along with computational methods and results. Examples of how Renewal Theory is best used are discussed.

CONTENTS

I. INTRODUCTION

This paper brings together the various scattered results on the Renewal Process model which are most relevant to applications. The early sections are expository in nature, describing the model and its applications, with particular emphasis on the Poisson process. Subsequent sections are devoted to general bounds and approximations for the renewal intensity, renewal function and forward recurrence distribution. A few special cases are worked out exactly and a numerical method for the discrete case is given. Special approximations for the Weibull case are included, and bounds on higher moments of the renewal counting process are given. Numerical illustrations are given whenever helpful.

A theoretical development of the Renewal Process model is found in [Smith, 1958]. The entire subject matter, including some extentions, is fully discussed there and an extensive bibliography is provided. A good introduction to the subject is [Cox, 1962], where the fundamentals are treated in a very clear manner, for the case where inter-event times have a density. An excellent discussion of Renewal Theory is in [Feller, 1971], where a section is devoted to the two-sided case as well. We will not consider the more general theory based on two-sided distributions, as our applications call for the nonnegative case only. Some references to this generality are [Stone, 1965 and 1966]. Proofs of most results are not given here, but instead intuitive reasoning is provided to assist the reader's insight and comprehension. Finally, the problems of estimation of rates, tests for trends, and other issues of a statistical nature are left to [Cox and Lewis, 1966] for resolution. This book gives an excellent discussion of the statistical problems which arise in practice.

II. POINT PROCESS

A point process is a series of epochs, $t_1 < t_2 < t_3 \ldots$ in time, which describe the evolution of a process, usually stochastic in nature. This model is used to describe such processes as failure epochs of complex equipments, emissions of particles during decay or pulses along a nerve fibre for example. The renewal process is a special kind of point process. If the interevent times $t_1, t_2 - t_1, t_3 - t_2, \ldots$, are statistically independent random variables, and all except possibly the first have a common probability distribution, we have a renewal process.

The simplifying assumptions of a renewal process make it a frequent choice as a model for certain point processes. One renewal process is most often assumed and its properties are particularly worth discussing, namely the Poisson process. It is the standard or benchmark to which other point processes are compared.

III. THE POISSON PROCESS

Consider a point process whose events occur at the epochs denoted by $0 < t_1 < t_2 < t_3 \ldots$. If $X_1 = t_1, X_2 = t_2 - t_1, X_3 = t_3 - t_2, \ldots$ are statistically independent random variables each with the exponential distribution,

$$\text{Prob}\{X_n \leq x\} = 1 - e^{-\lambda x}, \; x \geq 0 ,$$

then we call the process a (homogenous) Poisson process. The X_n are called interevent times and the parameter λ, the reciprocal of the mean (average) time between events, is called the rate of the process, as it is the mean number of events occurring per unit time over any time interval. In fact, the actual number of events occurring per unit time, over an increasing time period, tends almost surely to λ .

There are many physical processes for which evidence suggests the Poisson process as an appropriate model. These include telephone calls made during a particular hour of the day, emissions from a radioactive source over a period during which the strength of the source remains substantially constant, the occurrence of nerve pulses along a nerve fibre, or the failures of complex equipment during its operating periods. See in particular [Drenick, 1960] for some general conditions under which the failures of complex equipment form approximately a Poisson process. Also, [Cox & Lewis, 1966, p. 251] show some data from several physical processes which exhibit the Poisson process assumptions. The following description is particularly suggestive of its wide applicability.

Consider an arbitrary but fixed time horizon T , and a population of size n of individuals who may place a call during $(0, T]$. Suppose further that each individual decides to make a call with probability p , and that all individuals act independently, so that the number calling in $(0, T]$ has the Binomial distribution with parameters n, p . Assume further that those calling select a time to call independently of others and uniformly from $(0, T]$. The number placing a call in, say $(0, t]$, with $t \leq T$, has the Binomial distribution with parameters n , $p(t/T)$. Letting this random variable be N_t , we note that

$$E(N_t) = \lambda t$$

where

$$\lambda = np/T \ ,$$

and E denotes expectation. Now let n approach infinity and p approach zero such that λ , the mean number of calls per unit time, remains constant. The limiting process is a Poisson process. The distribution of N_t approaches the Poisson, mean λt , and the times between successive calls are independent and have the exponential distribution. One can imagine that a large population of potential callers, each one of whom has a very small probability of calling during $(0, T]$, and all of whom act independently, would approximate this limiting case quite well. The Poisson process is commonly referred to as the model for complete randomness in a point process.

Another description of the Poisson process in terms of N_t will help to illustrate the model. Note that N_t , the number of events during $(0, t]$, has a Poisson distribution, as this is the limit distribution for a Binomial with increasing n , decreasing p and constant mean np . Define $N_{t,t+h}$ to be the number of events occurring in $(t,t+h]$, so that

$$N_{t,t+h} = N_{t+h} - N_t \ .$$

The following assumptions also characterize the Poisson process.

(a) $N_{t,t+h}$ has the Poisson distribution with parameter λh , for all positive t and h .

(b) $N_{t,t+h}$ is statistically independent of the number and position of the events in $(0, t]$, for all positive t and h.

Some immediate consequences of (a) and (b) are:

(1) Prob $\{N_{t,t+h} = n\}$ = Prob $\{N_h = n\} = e^{-\lambda h}(\lambda h)^n/n!$ regardless of t, so that the process is stationary in time.

(2) Events occur singly since Prob$\{N_h \geq 2\} \sim (\lambda h)^2/2$ near $h = 0$.

(3) In a short interval of length h, the probability of one event occurring is approximately λh, and of no events, $1 - \lambda h$.

(4) The process' history prior to time t and its evolution after t are statistically independent.

From (1) it follows that the first event occurs after h with probability $e^{-\lambda h}$, so that X_1, the time to the first event, has the exponential distribution. This reasoning applies to successive interevent times X_n by conditioning on the value of t_{n-1}, so all interevent times have the exponential distribution. The independence of these times also follows from (4).

To see that our definition is consistent with assumptions (a) and (b), we note that

$$\{N_t < n\} = \{t_n > t\} .$$

This equivalence relates the counting variable N_t to the epochs t_n and implicitly to the interevent times X_n since $t_n = X_1 + .. + X_n$. For the Poisson process, the X_n are iid with the exponential distribution, so t_n has the Gamma distribution, yielding

$$\text{Prob}\{N_t < n\} = \text{Prob}\{t_n > t\} = \sum_{k=0}^{n-1} e^{-\lambda t}(\lambda t)^k/k! \; .$$

It follows that N_t has the Poisson distribution, with mean λt . The memoryless property of the exponential distribution is used to show (a) and (b) hold [Ross, 1972, p. 118].

The Poisson process enjoys many properties not shared by other point processes. These properties account in part for the wide use of the Poisson process in modelling problems. The counting variable $N_{t,t+h}$, the number of events in (t,t+h] , is of primary interest. It has the Poisson distribution with mean λh and becomes nearly normal as λh becomes large. By using a mean of λh and standard deviation of $\sqrt{\lambda h}$, the normal approximation is quite reasonable when $\lambda h \geq 20$. The counting variables corresponding to any set of disjoint intervals are statistically independent. This is particularly useful when accounting for the effects of events in different time periods, as the joint distribution of counts is easily specified.

Any (non-random) number of independent Poisson processes can be superimposed on a common time axis. The result is a Poisson process whose rate is the sum of the rates of the contributing processes. This follows directly from the reproductive property of the Poisson distribution under summation. It allows one to combine several processes without sacrificing any properties. In fact, when many independent non-degenerate point processes, not necessarily Poisson, are combined, the pooled process usually tends to a Poisson process. This fact also accounts for the general acceptance of the Poisson process when modelling the output of several independent processes of unknown origin. See [Cox & Smith, 1953, 1954] or [Khintchine, 1960] for a general discussion of the superposition of many processes and of its approach to the Poisson process.

Another operation under which the Poisson process is invariant is the random deletion of events. Suppose that every event is subject to deletion with probability $1 - p$, independent of other deletions. The remaining events form a Poisson process with rate $p\lambda$, where λ was the original rate. This result follows by either showing that the new interevent times are geometrically compounded exponentials and hence are still exponential, or by showing properties (a) and (b) hold by simple conditional probability arguments applied to the counting variable. This situation would arise if, for example, the original process could not be monitored directly, such as suicides from a bridge, or if one only recorded certain events, such as certain colored cars in a traffic stream; the recorded events would remain a Poisson process. As in the case of superposition, stationary point processes which are thinned this way become indistinguishable from Poisson Processes. By normalizing so the mean interevent time is one, the characteristic function for interevent times of the thinned process tends to $1/(1-\theta)$, continous at the origin, as the probability p of event observation tends to zero. This corresponds to the exponential probability distribution with unit mean. Hence, series of events with substantial random loss should follow a Poisson process approximately.

The epoch t_n at which the n^{th} event occurs is also of interest. As noted earlier, t_n has a gamma distribution with mean n/λ and variance n/λ^2 . This is also equivalent to the fact that $2\lambda t_n$ has the chi-squared distribution with $2n$ degrees of freedom, so standard tables can be used to find percentiles. This fact also provides a confidence limit or interval for λ based on observation of t_n . When n becomes large, the central limit theorem shows that $(\lambda t_n - n)/\sqrt{n}$ has the standard normal distribution approximately. For improved accuracy, note that $\sqrt{\lambda t_n}$ is nearly normal with mean $\sqrt{n-1/4}$ and variance 1/4.

The conditional joint distribution of $(t_1,\ldots,t_n)$, given the event $\{N_t = n\}$, is the same as the distribution of the order statistics from n independent samples of a uniform distribution over $[0,t]$. This means that by conditioning on the number of events in an interval, the event positions in the interval have a relatively simple probabilistic law. Several statistical tests for deviations from the Poisson process are based on this [Epstein, 1960]. Ross [1970, p. 18] uses this property to show that in a queue with an unlimited supply of servers and Poisson arrivals, the number of busy servers at time t is distributed Poisson with mean $n(t)$,

$$n(t) = \lambda \int_0^t (1-G(x))dx ,$$

where λ is the arrival rate and G is the distribution of service times. Since for large t the integral becomes $E(S)$, the mean service time, the steady state solution has the number of busy servers distributed Poisson with mean $\lambda E(S)$. This model may arise in the design of service systems where the number of servers is to be determined. The unlimited server case suggests the actual number needed for a low delay response. It may also arise where service is unlimited, such as for modelling the number recovering from industrial accidents.

The local variables of a point process refer to those variables measuring the process at the time t. For the Poisson process, these variables have simple laws. The forward recurrence time, Y_t, is the time from t forward to the next event. Since Y_t exceeds h when $N_{t,t+h} = 0$, it follows that Y_t has the exponential distribution with parameter λ. The backward recurrence time, Z_t, is the time since the last event prior to t, or t if none has occurred. By the same reasoning, Z_t has the exponential distribution with parameter λ but is truncated at t. From property (b), Y_t and Z_t are independent, so that the span, $S_t = Y_t + Z_t$ has, for $t >> \lambda^{-1}$, the gamma distribution with mean $2\lambda^{-1}$. This means that the interevent time which extends across the fixed time t is not a typical interevent time, as it has twice the mean of usual interevent times and a different distribution. This is a potential source of confusion and error in modelling problems, and it should be recognized at the outset.

A Poisson process in multiple dimensions is called spatial in [Feller 1968, p. 159]. He gives several examples of this case with data.

IV. RENEWAL PROCESSES

When a Poisson process is generalized by allowing the interevent times to have a distribution distinct from the exponential, the result is a renewal process. To fix the notation, we use t_n , $0 \le t_1 \le t_2 < \ldots$, as the time of the n^{th} event, or renewal. The interevent times are $X_n = t_n - t_{n-1} \ge 0$, which are taken to be independent random variables. It is sometimes convenient to allow X_1 to have a distribution distinct from the others; however, for now we assume not, so that

$$\mathrm{Prob}\{X_n \le x\} = F(x) \;, \quad \text{all} \;\; n \;,$$

and

$$\mathrm{Prob}\{X_n > x\} = \bar{F}(x) \;.$$

We also assume that F is a non-lattice distribution, except in the section on numerical methods for the discrete case. The second moment of F is assumed to be finite, with

$$\lambda^{-1} = \int_0^\infty x dF(x)$$

and

$$\sigma^2 = \int_0^\infty x^2 dF(x) - \lambda^{-2} \;.$$

In subsequent formulas, the product $\lambda\sigma$, referred to as the coefficient of variation c , appears as a correction factor to the exponential case. It makes a convenient and reasonable measure of the variability of a renewal process.

As before, we will use N_t as the number of events in (0,t] or

$$N_t = \max\{n | t_n \le t\} \;.$$

For problems involving $N_{t,t+h} = N_{t+h} - N_t$, we can refer to a renewal process whose first interevent time is Y_t , the forward recurrence time, as demonstrated later.

From the central limit theorem applied to the sums t_n , and the equivalence

$$\{N_t < n\} = \{t_n > t\} ,$$

we have that, as $t \to \infty$,

$$\frac{N_t - \lambda t}{\lambda\sigma\sqrt{\lambda t}} \xrightarrow{d} \text{Normal } (0,1) .$$

This means that, while the distribution of N_t usually is intractable, a normal approximation for large t exists, and standard tables can be used to obtain percentiles. Note that the coefficient of variation appears as a correction factor to the standard deviation for the Poisson case.

A quantity of considerable interest in modelling with a renewal process is the renewal function.

$$M(t) = E(N_t) .$$

This function, always non-negative and non-decreasing, appears in most formulas and is equivalent in information content to the distribution F . Both functions share common properties of differentiability, so that if F has a density, f , then M has a derivative, m , for which

$$M(t) = \int_0^t m(x)dx ,$$

where m, the renewal intensity, may involve impulse terms if f does.

A preliminary limiting result for the renewal function is that

$$M(t) = \lambda t + o(t)$$

$$\text{as} \quad t \to \infty ,$$

where $o(t)$ denotes a quantity tending to zero when divided by t. The only renewal process with a common interevent distribution for which $M(t) = \lambda t$ is the Poisson process. Under some regularity conditions (see [Smith, 1962]), the analogous result for the renewal intensity holds, namely

$$m(t) = \lambda + o(1)$$

$$\text{as} \quad t \to \infty ,$$

where $o(1)$ denotes a quantity tending to zero. Both of these results are intuitively suggested by the definitions.

For convenience, we will assume that the density f and renewal intensity m exist; however the methods illustrated do not depend upon this. There are two lines of reasoning which can be identified that work well when modelling with a renewal process. The first involves conditional expectation methods and the second an interpretation of the renewal intensity. We will illustrate both methods, as they often complement each other. The "renewal equation" on which some treatments of renewal theory are based, is discussed first (see [Feller, 1941]).

To derive an equation for the renewal function, M, first condition on X_1, the time of the first renewal. We have that

$$E(N_t | X_1 = x) = \begin{cases} 1 + M(t-x) & x \leq t , \\ 0 & x > t . \end{cases}$$

Unconditioning by the distribution of X_1, we see that

$$M(t) = E(N_t) = F(t) + \int_0^t M(t-x)f(x)dx \tag{1}$$

and by differentiation

$$m(t) = f(t) + \int_0^t m(t-x)f(x)dx . \tag{1a}$$

We will refer to this as a renewal equation; several other equations which follow appear different but are equivalent. By taking the Laplace transform of this equation, (transforms being denoted by an asterisk*) we have

$$M^* (s) = \frac{f^*(s)}{s(1-f^*(s))}$$

or equivalently

$$m^* (s) = \frac{f^*(s)}{1-f^*(s)} ,$$

where for any g, $g(t) = 0$ for $t < 0$, $g^*(s) = \int_0^\infty e^{-st} g(t)dt$.

This equation serves to show that the renewal function and the interevent distribution are equivalent in information content, and hence suggests that equations involving a renewal process can be expressed with the renewal function. As we shall have occasion to use results for the case where X_1 has a distribution different from the other interevent times, we leave as an exercise for the interested reader to show that

$$M_1(t) = F_1(t) + \int_0^t F_1(t-x)m(x)dx \ , \tag{1b}$$

where quantities for the general case are subscripted by 1.

The conditional expectation approach can be used to obtain the distribution of forward recurrence time. As earlier, we take Y_t to be the time elapsed from t until the next event, or $t_n - t$ for $n = N_t + 1$. Defining

$$G(y,t) = \text{Prob}\{Y_t \le y\} \ ,$$

we have by conditioning on the time until the first renewal

$$\text{Prob}\ \{Y_t \le y \mid X_1 = x\} = \begin{cases} G(y,t-x) & x \le t \ , \\ 1 & t < x \le t + y \ , \\ 0 & t + y < x \ . \end{cases}$$

After unconditioning, we have

$$G(y,t) = \int_0^t G(y,t-x)f(x)dx + F(t+y) - F(t) \ ,$$

which after some manipulation using the renewal equation (1) and transforms takes the form

$$G(y,t) = \int_t^{t+y} F(t+y-x)m(x)dx \ . \tag{2}$$

To illustrate the alternative method of derivation mentioned above, observe that $m(x)\delta x$ is, to first order in δx, $M(x+\delta x) - M(x)$, the mean of $N_{x,x+\delta x}$. As $\delta x \to 0$, this mean is $\text{Prob}\{N_{x,x+\delta x} = 1\} + o(\delta x)$.

In essence, the probability of multiple renewals in $(x,x+\delta x]$ is of order $(\delta x)^2$, so at most one renewal need be accounted for and the mean number to occur is asymptotically the probability of exactly one renewal in $(x,x+\delta x]$, as $\delta x \to 0$. Hence, $m(x)\delta x$ can be viewed as the probability of a renewal "at x". Equation (1a) can now be reasoned by seeing that $m(t-x)\delta t$ is the (conditional) probability of some renewal occurring at t , given the first renewal occurs at about x. From the law of total probability, and by passage to the limit in the choice of subintervals for x , we have that $\int_0^t m(t-x)f(x)dx \cdot \delta t$ is the probability of some renewal other than the first occurring at about t , to first order in δt. Now as $f(t)\delta t$ is the probability of the first renewal occurring at about t , to first order in δt , we have

$$m(t)\delta t = \left(f(t) + \int_0^t m(t-x)f(x)dx\right)\delta t + o(\delta t) ;$$

upon dividing by δt and letting $\delta t \to 0$, equation (1a) follows.

Equation (2) is obtained in a similar manner, by observing that $F(t+y-x)m(x)\delta x$, to first order, is the probability that the last renewal prior to $t+y$ occurs at about x , and that this event occurs at some x beyond t exactly when $Y_t \le y$. The case in which no density for the interevent distribution exists is handled by using the Lebesgue - Stieltjes integral and by replacing $m(x)dx$ by $dM(x)$.

Obtaining values for the distribution $G(y,t)$ depends on being able to obtain the renewal intensity. In a later section, some approximations are discussed. Here, the steady state result for large t is given:

$$G(y,t) \to F_e(y) = \lambda \int_0^y \bar{F}(x)dx \tag{3}$$

$$\text{as} \quad t \to \infty ,$$

where F_e is called the equilibrium distribution corresponding to F. This result can be obtained from equation (2) by seeing that, for large t, $m(x) \cong \lambda$ in the range of integration $(t,t+y]$. Replacing $m(x)$ by λ in (2), we see that, as $t \to \infty$,

$$G(y,t) \cong \int_t^{t+y} \bar{F}(t+y-x)\lambda dx = F_e(y) .$$

A rigorous limiting arguement is the Key Renewal theorem which is proven in [Smith, 1958].

When the renewal process is first observed at a time t from its actual origin, the first interevent time is Y_t. For many practical situations, one must assume t is large enough to use steady state results for determining the distribution of Y_t. If we let X_1 have the equilibrium distribution F_e for F, while all the remaining interevent times have distribution F, then we have that

$$M_e(t) = F_e(t) + \int_0^t F_e(t-x)m(x)dx ,$$

where $M_e(t)$ is the mean number of renewals in $(0,t]$ for this equilibrium renewal process. Taking the Laplace transform of this equation gives

$$M_e^*(s) = F_e^*(s) + F_e^*(s)m^*(s)$$

and since

$$F_e^*(s) = \frac{\lambda}{s^2}(1 - f^*(s)) ,$$

we have $$M_e^*(s) = \frac{\lambda}{s^2} ,$$

or
$$M_e(t) = \lambda t .$$

Thus for the renewal process which is observed in steady state, the renewal function is λt .

The backwards recurrence time distribution can be obtained in the same manner as in the forward recurrence case. In particular, the same limiting distribution is obtained;

$$\text{Prob}\{Z_t \leq z\} \to F_e(z)$$

$$\text{as} \quad t \to \infty ,$$

where Z_t is the time since the last renewal prior to t . Note that the exponential distribution is its own equilibrium distribution, and is unique in this respect. Finally, we note that our previous use of the renewal intensity immediately yields $\bar{F}(t-z)m(z)$ as the density of Z_t in $(0,t)$, with probability mass $\bar{F}(t)$ at $z = t$. Integrating this density gives

$$1 = \bar{F}(t) + \int_0^t \bar{F}(t-z)m(z)dz \tag{4}$$

which is equivalent to the renewal equation (1).

The forward and backward recurrence times sum to the span, S_t , which is the length of the interevent interval intersecting t . As noted in the Poisson process case, the span does not have the interevent distribution F . The sampling of interevent intervals is biased toward the longer intervals, as these are more likely to intersect the fixed time t . If $h(s,t)$ is the density of S_t , then for $s < t$, the probability $h(s,t) \cdot \delta s$

is the probability of a renewal occurring at x and the subsequent interevent interval being of length s , summed over x in $[t-s,t]$, or

$$h(s,t) = \int_{t-s}^{t} f(s)m(x)dx = f(s)[M(t)-M(t-s)] .$$

As t tends to infinity, $h(s,t)$ becomes $h(s)$, where

$$h(s) = \lambda s f(s) ,$$

the steady-state density of span. This result is obtained by observing that $M(t) \cong \lambda t$ for large t . The result is actually an application of Blackwell's theorem, [Blackwell, 1948] . The limiting result can also be obtained by arguing that in a set of n random intervals, $X_1,\ldots,X_n$, the combined expected length of all those of size x such that $|x-s| < \frac{\delta s}{2}$, is approximately $n \cdot s \cdot f(s) \cdot \delta s$ which normalized by $\sum_{i=1}^{n} X_i$ is the probability of a span being within δs of s . As n becomes large, we have the limiting result derived above for the asymptotic density of span by the strong law of large numbers.

The remaining paragraphs of this section are devoted to the description of examples of applications involving the renewal process model.

An early application of the model is to the theory of counters, as in [Feller, 1948]. A Geiger-Müller counter registers the emission of particles from a decaying substance, however due to the counter's resolving time, not all events are recorded. The evolution of recorded events can be modelled as a renewal process from which an estimate of the original process parameters can be made.

Models for a reliability problem frequently involve a renewal process. See [Gnedenko et. al., 1969] for a treatment of reliability theory using renewal theory. The interevent intervals correspond to machine operating periods and the renewal epochs to machine failures. The tail distribution for forward recurrence time, $\text{Prob}\{Y_t > y\}$, is the interval reliability as defined in [Barlow and Proschan, 1965, p. 8]. A model for the single item failure and repair process is the alternating renewal process. In this model, the process is in one of two states, 0 and 1 , and alternates between them. Residence times in state 0(1) are identically distributed samples from a distribution $F_0(F_1)$, and all residence times form an independent set. The epochs at which the process enters state 0 , or state 1 , form two imbedded renewal processes. The interevent distribution for either imbedded process is simply the convolution of F_0 with F_1 . To find the availability function, $A(t)$, which is the probability that the process is in, say state 1 at time t , assuming it began in state 1, we have that

$$A(t) = \bar{F}_1(t) + \int_0^t [\bar{F}_1(t-x)]m(x)dx \ .$$

The reasoning is that either state 1 was not left in $[0,t]$ or, after reentering state 1 at time x , the process remained in state 1 beyond time t , for some time x in $[0,t]$. Some bounds in special cases are given in [Butterworth, 1963]; exact solutions are usually not available. Approximations can be found by using one of the approximations to $m(x)$ reported below. The steady state availability follows immediately by replacing $m(x)$ by λ , its relevant value in the integrand when t is large, to obtain $\mu_1/(\mu_1 + \mu_0)$ in the limit. Here μ_i is the mean residence time in state i. The interval availability, $A(t,t+h)$ defined as the probability of continued residence in state i over the interval $[t,t+h]$ for an alternating renewal process, has an analogous derivation leading to

$$A(t,t+h) = \bar{F}_1(t+h) + \int_0^t [\bar{F}_1(t+h-x)]m(x)dx \ . \tag{5}$$

The steady state value in this case is just $[\mu_1/(\mu_1 + \mu_0)]\ [\bar{F}_{1e}(h)]$, which is suggested by the form of the expression.

The study of replacement policies for failing equipment may involve the renewal model. Several bounds on the renewal functions are determined in [Barlow and Proschan, 1964] (and by alternate methods in later sections here) by comparison of certain replacement policies. In particular, for a block replacement policy in which items are replaced every b units of time, and at failure, the expected number of failures between preventive replacements is $M(b)$. Accurate estimates of this may require knowledge of the renewal function for relatively short times b. The approximations given below are particularly directed toward the behavior of the renewal function near the origin.

Bartholomew [1959, 1963[b]] applies the renewal model to problems in labor turnover, especially in circumstances where steady state-results are not of interest. For example, the renewal intensity is interpreted as the labor turnover rate for new organizations, and the interevent distribution as a means of measuring stability. An approximation to the renewal intensity for its transition from the origin to its steady state value is necessary for predicting the turnover effect. Actual steady state doesn't occur within the period of model applicability.

The queueing theory literature relies on the renewal process model to describe the input process to many service systems. Marshall and Wolff [1971] use bounds on the renewal function to get bounds on the difference between customer average and time average measurements for some queues. A related field, Inventory Theory, has problems involving a renewal process in the evaluation of the $s - S$ policy (see [Arrow, Karlin & Scarf, 1958]).

The following sections will give some bounds and approximations with practical significance, and briefly review some related literature.

V. THE RENEWAL FUNCTION

As we stated in section IV, a quantity which continually arises in renewal theory, and in the analysis of many models in which renewal processes are imbedded, is the renewal function, the expected number of renewals which occur in the interval $(0,t]$, considered as a function of t. We defined this to be $M(t)$ for an ordinary renewal process, and showed that

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx \ , \qquad t \geq 0 \ . \tag{1}$$

Another important representation of $M(t)$ is obtained by noting that the counting random variable $(N_t + 1)$ is a stopping time in the sense of Wald, and by looking at the time to the first event after t, that is $t + Y_t$, after taking expectations

$$M(t) = \lambda t + \lambda y(t) - 1 \ , \qquad t \geq 0 \ , \tag{6}$$

where $y(t) = E[Y_t]$.

a) Approximations for the Renewal Function

For all but a few simple processes it is difficult to solve equation (1) for $M(t)$. It is equally difficult to determine $M(t)$ in other ways, such as from (6), but since the renewal function appears in so many expressions for other quantities of interest, it is important that we can at least find approximations to $M(t)$. Early results emphasized asymptotic theory for large t . Clearly $M(t) \geq 0$ and it is non-decreasing. The simplest non-trivial result, often called the elementary renewal theorem (see, for example [Cox, 1962]) is that for any renewal process with finite mean interevent time $1/\lambda$,

$$\lim_{t \to \infty} \frac{M(t)}{t} \to \lambda ,$$

so that λ can be interpreted as the "rate" of the process.

Clearly $M(t)$ increases "on the average" at rate λ . The next result, which can be stated in a number of forms, shows that $M(t)$ approaches a linear function with slope λ (recall we are assuming that F is non-lattice. When F is lattice, consider $M(t)$ to be defined only at multiples of the lattice; then $M(t)$ is asymptotically linear). This well-known result is known as Blackwell's Theorem, and it states that for any $h > 0$, with the restrictions just discussed,

$$\lim_{t \to \infty} [M(t+h) - M(t)] = \lambda h .$$

A similar result is found in [Smith, 1957] . Recall that σ^2 is the variance of the inter-event times, and $c^2 = \lambda^2\sigma^2$, the coefficient of variation squared. Then if either F is non-lattice, or if t is restricted to points on the lattice, then

$$M(t) = \lambda t + \frac{c^2-1}{2} + o(1) .$$

Let

$$A_1(t) = \lambda t + \frac{c^2-1}{2} . \tag{7}$$

In this paper $A_i(t)$ will be used to denote the ith approximation to $M(t)$. Equation (7) can be used as an approximate formula for $M(t)$. It can be quite accurate for "large" t, where "large" depends on the distribution F as well as its mean. For many distributions, for $t = 4$ or 5 multiple of the mean, (7) is quite accurate. However if F is highly skewed, convergence to the linear function can be very slow.

Equation (7) has certain advantages and disadvantages. First, it is linear, easy to compute, and one needs knowledge of only the mean and variance of the interevent time. To offset these advantages, it can be quite poor for approximating $M(t)$ for small t, and it is not known whether the true $M(t)$ lies above as below $A_1(t)$ for any given t.

Let us write $A_1(t)$ as $\lambda t + k$, where $k = \frac{c^2-1}{2}$. Now substitute this approximation to $M(t)$ into the right hand side of the renewal equation (1). After a little algebra the expression simplifies to $A_2(t)$, where

$$A_2(t) = \lambda t + (1+k)F(t) - F_e(t) . \tag{8}$$

This gives us a second approximation for $M(t)$. It is easy to compute and has the correct asymptotic behavior; however, we have lost the linear property. We are assured that $A_2(t)$ is non-negative. This last property is seen by noting that $(1+k) \geq 0$, and $F_e(t) = \lambda \int_0^t \bar{F}(u)du \leq \lambda t$.

From equation (1) it is easy to derive the following expression for $M(t)$ either directly or by transform methods,

$$M(t) = \lambda t + \int_0^t \bar{F}_e(t-u)m(u)du - F_e(t) . \tag{9}$$

Equation (9) is useful for determining another approximation to $M(t)$. Let us replace $m(u)$ on the right hand side of (9) by λ, an exact expression for the case of a Poisson Process, but in general an approximation. Then

$$A_3(t) = \lambda t + \lambda \int_0^t \bar{F}_e(u)du - F_e(t) \tag{10}$$

is an approximation to $M(t)$. Like $A_2(t)$ it is no longer linear, and it requires knowledge of F and not just its moments. It is easy to show that

$$\lim_{t \to \infty} (A_2(t)-\lambda t) = \frac{c^2-1}{2} ,$$

so that the approximation is asymptotically correct.

The next approximation is derived from an approximation to the renewal intensity $m(t)$ derived in [Bartholomew, 1963]. Although it does not appear that the integral of the approximation was intended to be used as an approximation to M, it appears to give very good results for small t, although it may be poor for large t. In the notation of this paper, Bartholomew shows that

$$m(t) \approx f(t) + \lambda \frac{F(t)^2}{F_e(t)} ,$$

where f is the density of F. After a little algebra, the integral of his approximation (which we call $A_4(t)$) can be written

$$A_4(t) = \lambda t + F(t) - \lambda \int_0^t \left(1 - \frac{F(u)^2}{F_e(u)}\right) du . \tag{11}$$

For many distributions it is computationally more difficult to use (11) than (10) or (8) because of the quotient in the integral. Note that the denominator F_e is itself an integral of $\bar{F}$.

Let

$$K(t) = \int_0^t \left(1 - \frac{F(u)^2}{F_e(u)}\right) du$$

$$= \int_0^t \frac{F_e(u) - F(u)^2}{F_e(u)} \cdot du .$$

Let

$$\Delta(t) = F_e(t) - F(t) .$$

Then

$$K(t) = \int_0^t \bar{F}(u)du + \int_0^t \Delta(u) \frac{F(u)}{F_e(u)} \cdot du .$$

Now if $F_e(t) > F(t)$ for some t, then $\Delta(t)\frac{F(t)}{F_e(t)} < \Delta(t)$.

Similarly if $F_e(t) < F(t)$ the same inequality holds. Note also that since $\Delta(t) = \bar{F}(t) - \bar{F}_e(t)$,

$$\int_0^\infty \Delta(u)du = \frac{1-c^2}{2\lambda} .$$

Thus unless $F_e(t) = F(t) \ \forall \ t \geq 0$, then

$$K(\infty) < \frac{3-c^2}{2\lambda} .$$

From (11) we see that

$$A_3(t) = \lambda t + F(t) - \lambda K(t) ,$$

and so if $F_e(t) \neq F(t)$ for some t, then

$$\lim_{t \to \infty} [A_3(t) - \lambda t] > \frac{c^2 - 1}{2} - 1 .$$

The approximations given so far are not restricted to any particular family of distributions. Papers have appeared where F is restricted to a particular class. Leadbetter (1963] proves the following result. Suppose we can write F as an absolutely convergent power series

$$F(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\Gamma(1+mn)} c_n t^{mn} , m > 0 .$$

Define

$$D_1 = C_1$$

$$D_2 = C_2 - D_1 C_1$$

$$\vdots$$

$$D_n = C_n - \sum_{j=1}^{n-1} D_j C_{n-j} .$$

Then

$$M(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\Gamma(1+mn)} D_n t^{mn} , \quad m > 0 , \tag{12}$$

where the series converges absolutely for all $t \geq 0$.

Applications of (12) to the case when F has a Weibull distribution are given in [Leadbetter, 1963] and [Smith and Leadbetter, 1963] . In the cases treated, only the first four terms of (12) were required. It should be mentioned that they treat higher moments of N_t and in fact plot the variance of N_t for various values of the parameter in the Weibull distribution. We do not pursue higher moments in this paper.

Further work on renewal processes for which F has a Weibull distribution is to be found in [Lomnicki, 1966]. He discusses both the distribution of the number of renewals and the renewal function. We quote only his approximations to the renewal function.

The basic idea in the work of Lomnicki is to write the renewal function as a power series whose terms are easy to calculate from available tabulated functions, and where convergence is rapid so that only a few terms need to be calculated.

Assume that the underlying distribution F is given by

$$F(t) + 1 - e^{-t^{\beta}} , \qquad t \geq 0 , \qquad \beta > 0 .$$

Define $D_k(t) = e^{-t} \sum_{r=k}^{\infty} \frac{t^r}{r!} , \qquad k = 1,2,3, \ldots \ .$

The functions D are tabulated as the tail distribution of a Poisson random variable with mean t. Now define

$$\gamma(r) = \frac{\Gamma(\beta r-1)}{\Gamma(r+1)} , \quad r, 0, 1, 2, \ldots ,$$

where β is the Weibull parameter. Next define

$$b_0(s) = \gamma(s) , \ s = 0, 1, 2, \ldots$$

$$b_{k+1}(s) = \sum_{r=k}^{s-1} b_k(r)\gamma(s-r) , \quad k = 0, 1, \ldots \ldots ,$$
$$s = k+1, k+2, \ldots \ .$$

Next define

$$a_k(s) = \sum_{p=k}^{s} (-1)^{p+k} \binom{s}{p} \frac{b_k(p)}{\gamma(p)} , \quad k = 0, 1, 2, \dots ,$$
$$s = k, k+1, \dots .$$

Finally let

$$\alpha_k(k) = a_k(k) ,$$

$$\alpha_k(s) = \sum_{r=k}^{s} a_r(s) - \sum_{r=k}^{s-1} \alpha_r(s) , \quad s = k+1, k+2, \dots .$$

Then the renewal function

$$M(t) = \sum_{s=1}^{\infty} D_s(t^{\beta}) \sum_{k=1}^{s} \alpha_k(s) .$$

The interested reader should consult the Lomnicki paper for details such as convergence properties. The paper shows numerical calculations for a number of interesting functions in a Weibull renewal process.

b) Bounds for the Renewal Function

Instead of approximations of the type presented in a) it is often desirable to have upper and lower bounds on the renewal function, and a number of such bounds are presented here. It is usually easier to find lower bounds. Most upper bounds apply only with restricted classes of distributions, although some general upper bounds are given.

The simplest non-trivial lower bound on $M(t)$ is obtained from equation (6). By noting that $y(t)$ is the expectation of a non-negative random variable we have our first lower bound $L_1(t)$, where

$$L_1(t) = \lambda t - 1 . \qquad (13)$$

L_i will be used for the ith lower bound, and U_i for the ith upper bound.

A lower bound was derived in [Barlow and Proschan, 1965, p. 68] in an interesting way, by comparing block and age replacement policies in a reliability theory context. The same bound, however, is easily obtainable from equation (9) directly. Since $\bar{F}_e$ is a monotone non-increasing function, equation (9) gives

$$M(t) \geq \lambda t + \bar{F}_e(t)M(t) - F_e(t) . \tag{14}$$

Thus

$$M(t) \geq L_2(t) = \frac{\lambda t}{F_e(t)} - 1 . \tag{14a}$$

As was pointed out earlier, $F_e(t) \leq \lambda t$, and so $L_2(t) \geq 0$. It is clear also that since $F_e(t) \leq 1$, that $L_2(t) \geq L_1(t)$.

Equation (1) can be used to find a simple upper bound on $M(t)$. Since F is monotone non-decreasing it is easy to see from (1) that

$$M(t) \leq U_1(t) = \frac{F(t)}{\bar{F}(t)} . \tag{15}$$

In general this bound is very poor and is of little practical use. The search for a good upper bound for a general process is more difficult. Stone [1972] in a short interesting paper derives the following very useful upper bound which has the nice properties of being linear in t and requiring only the first two moments of the underlying distribution;

$$M(t) < U_2(t) = \lambda t + 3(1+c^2) . \tag{16}$$

(Stone in this and a number of other papers deals with renewal processes in which the random variables can be negative. Since our motivation is taken from reliability, inventory, and queuing theory where the random variables are invariably non-negative we do not pursue this two-sided generalization.)

If one is willing to restrict the class of distribution functions, other interesting bounds can be obtained. Many classes of distributions are described in [Marshall and Proschan, 1970], a paper which studies the relationship between various classes.

Let us assume that the interevent distribution has the property

$$\int_t^{\infty} \frac{\bar{F}(u)du}{\bar{F}(t)} \leq 1/\lambda \ , \qquad \text{all } t \geq 0 \ . \tag{17}$$

The left hand side of this inequality can be thought to represent the mean residual lifetime of a component which has been in service a time t and which has a random lifetime distributed as F with unconditional mean $1/\lambda$.

In reliability theory terminology we say that F belongs to the class of NBUE distributions (New-Better-than-Used-in-Expectation), since clearly the inequality says that a used component can never have a mean residual life longer than the mean life of a new component. Any distribution for which the inequality in (17) is reversed is called UBNE with the obvious connotation. Equation (17) can also be written as

$$\bar{F}_e(t) \leq \bar{F}(t) \ , \qquad \text{all } \ t \geq 0 \ . \tag{17a}$$

Equation (1) can be re-written in the form

$$F(t) = \int_0^t \bar{F}(t-u)m(u)du \ . \tag{18}$$

Now if we assume that F is NBUE(UBNE) then using (17a) and (18) in (9) shows that

$$M(t) \underset{(\geq)}{\leq} \lambda t \ . \tag{19}$$

In this paper acronyms (such as UBNE) in parentheses should be read together with the inequalities in parentheses. Equations (13) and (19) together show that for F an NBUE distribution

$$\lambda t - 1 \leq M(t) \leq \lambda t \ , \tag{20}$$

and we have very simple linear bounds. If F is UBNE (13) can be improved to $M(t) \geq \lambda t$.

A stronger assumption on F is to say that F has increasing failure rate (IFR). For details of IFR or other classes the reader should consult [Barlow and Proschan, 1965] or[Marshall and Proschan, 1972]. Here we shall say that F is IFR if log $\bar{F}(t)$ is concave in t . For a renewal process with F restricted to this class, Barlow and Proschan [1965, p. 71], show that

$$M(t) \leq \lambda t \frac{F(t)}{F_e(t)} = U_4(t) \ .$$

This result is obtained by comparing different replacement policies. No direct way of reproducing this result has been found.

The linear bounds in (2) are very useful in many applications. Some are discussed later in this paper. If each bound is iterated in the basic renewal equation (1) , the reader will find that

$$L_3(t) = \lambda t - F_e(t) \le M(t) \le \lambda t - F_e(t) + F(t) = U_3(t) \ .$$

It is interesting to note that the convex combination of these bounds which has the correct asymptotic behavior is the approximation $A_2(t)$ in equation (8) .

Continued iteration of either bound yields a sequence of bounds which monotonically converges on $M(t)$ from below or above. From a computational viewpoint this is an attractive property. The question naturally arises, can one start with better bounds than those in (20)? Of course the upper bound in (20) holds only for NBUE distributions, but Stone's upper bound could be used to start the iterative procedure.

Marshall [1973] studied the problem of improving the starting bounds in this iterative procedure. It is easy to show that for any constant, say x , if the function $\lambda t + x$ is used to start an iterative procedure in (1), the procedure will eventually converge on $M(t)$. Our problem is to find two numbers, b_ℓ and b_u such that

$$\lambda t + b_\ell \le M(t) \le \lambda t + b_a \ ,$$

where $\lambda t + b_\ell$ is as large as possible, and $\lambda t + b_u$ is as small as possible, and such that when either one is iterated in the renewal equation (1), an improved† bound is obtained for all t . These bounds are called the "best" linear bounds.

To illustrate the method we use the lower bound. A simple iteration of $\lambda t + b_\ell$ in (1) shows that the inequality

† By "improved", we mean a bound which is no worse than the one in question.

$$\lambda t + b_\ell \le \lambda t + (1+b_\ell)F(t) - F_e(t)$$

must hold if the improvement is taking place.

Thus

$$b_\ell \le \frac{\bar{F}_e(t)}{\bar{F}(t)} - 1 ,$$

and we make b_ℓ as large as we can. Let A be the set of t for which $\bar{F}(t) > 0$. Thus the reader should see that the largest b_ℓ and smallest b_u are given respectively by

$$b_\ell = \operatorname*{Inf}_{t\varepsilon A} \frac{\bar{F}_e(t)}{\bar{F}(t)} - 1 ,$$

and

$$b_u = \operatorname*{Sup}_{t\varepsilon A} \frac{\bar{F}_e(t)}{\bar{F}(t)} - 1 .$$

Since $\bar{F}_e(0^-) = \bar{F}(0^-) = 1$, $-1 \le b_\ell \le 0$. Also if F is NBUE, then $\bar{F}_e(t) \le \bar{F}(t)$, and so $b_u = 0$. In this case (19) is indeed the "best" linear upper bound. For some distributions b_u is not finite, in which case $M(t)$ cannot be bounded for all t by a linear function.

c) Examples and Calculations.

For the Poisson process (i.e., F is exponential) the solution of (1) is $M(t) = \lambda t$. In this section (1) is solved for a number of renewal processes and compared to the approximations and bounds discussed in V a) and b).

The first case considered is when F has a uniform distribution over $(0, 2/\lambda)$. For this case (1) can be solved for successive intervals of length $2/\lambda$, and we find

$$M(t) = \sum_{n=0}^{j-1} \frac{1}{n!} (n - \frac{\lambda t}{2})^n e^{\frac{\lambda t}{2} - n} - 1 ,$$

$$2(j-1)/\lambda \leq t \leq 2j/\lambda,$$

$$j = 1, 2, 3, \ldots \quad .$$

Note that the uniform distribution is in the NBUE class, and it is easy to see that the best linear bounds are given by (20) . Table 1 shows some of the approximations and bounds for the case $\lambda = 2$ for t up to 10 mean lifetimes.

The second case shown is when F has a gamma distribution with mean 1 and variance $\cdot 5$. As an example with $C^2 > 1$ a third case is shown with F a hyperexponential distribution with mean 1 and variance 2.5 . Calculations for the gamma and exponential cases are shown in tables 2 and 3 respectively. The gamma distribution is NBUE with $k = -.25$, $b = -.5$ and $b_u = 0$. The hyperexponential distribution is UBNE with $k = .75$, $b_\ell = 0$ and $b_u = 1.5$.

t	$L_3(t)$	$L_2(t)$	$U_3(t)$	$U_4(t)$	$A_4(t)$	$A_3(t)$	$A_2(t)$	$M(t)$
0.1	0.002	0.026	0.052	0.051	0.051	0.098	0.036	0.051
0.2	0.010	0.053	0.110	0.105	0.105	0.191	0.077	0.105
0.3	0.022	0.081	0.172	0.162	0.162	0.280	0.122	0.162
0.4	0.040	0.111	0.240	0.222	0.221	0.365	0.173	0.221
0.5	0.062	0.143	0.312	0.286	0.284	0.448	0.229	0.284
0.6	0.090	0.176	0.390	0.353	0.350	0.528	0.290	0.350
0.7	0.122	0.212	0.472	0.424	0.419	0.606	0.356	0.419
0.8	0.160	0.250	0.560	0.500	0.493	0.683	0.427	0.492
0.9	0.202	0.290	0.652	0.581	0.570	0.758	0.502	0.568
1.0	0.250	0.333	0.750	0.667	0.651	0.833	0.583	0.649
2.0	1.000	1.000	2.000	2.000	1.773	1.667	1.667	1.718
5.0	4.000	4.000	5.000	5.000	4.773	4.667	4.667	4.666
8.0	7.000	7.000	8.000	8.000	7.773	7.667	7.667	7.666
10.0	9.000	9.000	10.000	10.000	9.773	9.667	9.667	9.666

TABLE 1. $M(t)$ and approximations for the uniform distribution with mean 1.

t	$L_3(t)$	$L_2(t)$	$U_3(t)$	$U_4(t)$	$A_4(t)$	$A_3(t)$	$A_2(t)$	$M(t)$
0.1	0.001	0.006	0.018	0.018	0.018	0.096	0.014	0.018
0.2	0.004	0.022	0.066	0.063	0.063	0.185	0.051	0.062
0.3	0.013	0.047	0.135	0.128	0.126	0.270	0.105	0.125
0.4	0.029	0.078	0.220	0.206	0.203	0.352	0.172	0.200
0.5	0.052	0.116	0.316	0.295	0.289	0.434	0.250	0.284
0.6	0.082	0.158	0.419	0.391	0.381	0.516	0.335	0.373
0.7	0.119	0.205	0.527	0.492	0.477	0.598	0.425	0.465
0.8	0.163	0.257	0.638	0.597	0.576	0.681	0.520	0.560
0.9	0.214	0.312	0.751	0.705	0.677	0.766	0.617	0.657
1.0	0.271	0.371	0.865	0.814	0.779	0.852	0.716	0.755
2.0	1.055	1.116	1.963	1.922	1.807	1.773	1.736	1.750
5.0	4.000	4.001	5.000	4.999	4.821	4.750	4.750	4.750
8.0	7.000	7.000	8.000	8.000	7.822	7.750	7.750	7.750
10.0	9.000	9.000	10.000	10.000	9.822	9.750	9.750	9.750

TABLE 2. $M(t)$ and approximations for the gamma distribution with mean 1 and variance 1/2.

t	$L_3(t)$	$L_2(t)$	$U_3(t)$	$U_4(t)$	$A_4(t)$	$A_3(t)$	$A_2(t)$	$M(t)$
0.1	0.008	0.081	0.153	0.158	0.158	0.103	0.263	0.158
0.2	0.028	0.165	0.295	0.310	0.311	0.210	0.495	0.311
0.3	0.060	0.251	0.427	0.459	0.460	0.321	0.702	0.460
0.4	0.101	0.338	0.551	0.602	0.605	0.435	0.889	0.605
0.5	0.150	0.427	0.669	0.741	0.747	0.551	1.059	0.747
0.6	0.205	0.517	0.782	0.876	0.886	0.669	1.215	0.886
0.7	0.265	0.609	0.891	1.007	1.022	0.788	1.361	1.022
0.8	0.330	0.701	0.997	1.134	1.156	0.907	1.497	1.155
0.9	0.398	0.793	1.100	1.258	1.287	1.027	1.626	1.285
1.0	0.470	0.886	1.201	1.378	1.415	1.147	1.749	1.413
2.0	1.288	1.808	2.162	2.454	2.617	2.332	2.817	2.599
5.0	4.085	4.462	5.051	5.277	5.837	5.623	5.775	5.736
8.0	7.025	7.209	8.015	8.125	8.893	8.712	8.758	8.749
10.0	9.011	9.116	10.007	10.069	10.906	10.733	10.753	10.750

TABE 3. $M(t)$ and approximations for the hyperexponential distribution with mean 1 and variance 5/2.

VI. THE RENEWAL INTENSITY AND ITS CONVOLUTIONS.

The previous section gives approximations and bounds on the renewal function. In this section, some approximations for the renewal intensity are shown. Also some approximations to convolutions of tail distribution functions with the renewal intensity are given and some applications for these are discussed.

The renewal intensity function $m(t)$ gives the rate of renewals occuring at the instant t from the start of a renewal process. If many (say N) identical renewal processes are operating simultaneously, then $N \cdot m(t)$ is the rate of renewals for the composite process. This fact is at the basis of an application to labour turnover. A new organization of fixed size experiences a turnover in its labour force which has been observed to fit this renewal model. See [Bartholomew, 1959] for a complete discussion with some data.

As mentioned in an earlier section, $m(t)$ approaches under suitable regularity conditions, the rate λ for the process, as t tends to infinity. Our approximation is meant to describe the transition from $M(0) = f(0)$ to the final value of λ. Perhaps the best approximation we know of is the one given in the reference [Bartholomew, 1963[a]], which we call $h_1(t)$, where

$$h_1(t) = f(t) + \frac{\bar{F}^2(t)}{\int_0^t \bar{F}(u)du} \quad . \tag{21}$$

As seen later in numerical illustrations, h_1 is an excellent approximation to m, particularly for highly skew interevent distributions, which is exactly when approximations are most needed. This is explained in part by the observations given by Bartholomew which are repeated here. The renewal equation (4) derived in section IV is essentially

$$1 = \bar{F}(t) + \int_0^t m(t-z)\bar{F}(z)dz ,$$

or equalently

$$1 = F(t)/\int_0^t m(t-z)\bar{F}(z)dz .$$

From the renewal equation (1a) we also have

$$m(t) = f(t) + \int_0^t m(t-u)f(u)du . \tag{1a}$$

Upon combining these, this renewal equation may be written

$$m(t) = f(t) + F(t) \int_0^t m(t-u)f(u)du/\int_0^t m(t-u)\bar{F}(u)du .$$

The approximation h_1 then results by replacing the ratio

$$\int_0^t m(t-u)f(u)du/\int_0^t m(t-u)\bar{F}(u)du$$

by

$$\int_0^t f(u)du/\int_0^t \bar{F}(u)du .$$

Since $m(t) = \lambda$ a constant for the exponential case, it follows that $h_1 = m$ for a Poisson process. Further, in cases where F is highly skewed so that $m(t-u)$ in more nearly constant in u where $f(u)$ and $\bar{F}(u)$ have appreciable positive value, the factoring and cancellation of $m(t-u)$ is not so unreasonable and the approximation is close.

Other reasons for the success of h_1 are:

(1) $h_1(0) = m(0)$

(2) $\lim_{t \to \infty} h_1(t) = \lim_{t \to \infty} m(t) = \lambda$

(3) $h_1'(0) = m'(0)$ and $h_1''(0) = m''(0)$

where prime denotes differentiation with respect to the single variable t.

An advantage of h_1 is that knowledge of the interevent distribution F is necessary only up to t; that is, $h_1(t)$ is independent of the tail of F beyond t. This can be of considerable advantage when F is skewed and must be estimated from data. In particular, methods depending on transforms are inherently sensitive to the tail of F, a function which may not be well known in the upper regions, particularly if data is not available. The only possible drawback to h_1 is the appearance of $\int_0^t \bar{F}(u)du$ which might have to be obtained by numerical integration. Even this however, should not be difficult.

Another approximation to $m(t)$ to consider is $h_2(t)$ where

$$h_2(t) = f(t) + \lambda F(t) . \tag{22}$$

This approximation results by replacing $m(t-u)$ by λ in the right-hand side of equation (1a). Its relative lack of accuracy when compared to h_1 is somewhat compensated for by the simpler functional form, however the choice of which one to use is best left to those making the application. While h_2 can itself be substituted in the righthand side of (1a) to give an improved approximation, the appearance of convolutions quickly limits the practable usefulness of this approach, particularly in view of the availability of h_1.

Tables 4 & 5 shown below give an indication of the numerical accuracy involved with these approximations.

t	m(t)	$h_1(t)$	$h_2(t)$	m(t)	$h_1(t)$	$h_2(t)$
0.0	1.60	1.60	1.60	4.51	4.51	4.51
0.2	1.51	1.51	1.36	4.11	4.11	2.24
0.4	1.44	1.44	1.21	3.75	3.77	1.40
0.8	1.32	1.32	1.04	3.16	3.25	.99
1.2	1.23	1.24	.98	2.69	2.87	.93
1.6	1.17	1.18	.95	2.32	2.60	.93
2.0	1.12	1.14	.95	2.04	2.39	.93
2.5	1.08	1.11	.95	1.76	2.18	.94
3.0	1.05	1.08	.96	1.56	2.02	.94
5.0	1.01	1.03	.98	1.17	1.62	.95
10.	1.00	1.00	1.00	1.01	1.25	.97

$\lambda = 1.0$, $c = 1.58$ $\qquad$ $\lambda = 1.0$, $c = 3.54$

TABLE 4. Calculations using a Hyper-Exponential interevent distribution with the given rate λ and coefficient of variation c.

t	$m(t)$	$h_1(t)$	$h_2(t)$
0.0	0.	0.	0.
.01	.039	.039	.039
.02	.077	.077	.078
.04	.148	.148	.151
0.1	.330	.331	.345
0.2	.551	.556	.598
0.4	.798	.818	.910
1.0	.982	1.03	1.14
2.0	1.00	1.02	1.05
4.0	1.00	1.00	1.00

$\lambda = 1.0 \quad c = 0.77$

TABLE 5. Calculations using an Erlang interevent distribution with shape and scale parameters = 2.0 .

Another renewal process quantity to be approximated is a convolution of the form

$$c(t) = \int_0^t \bar{G}(t-u)m(u)du$$

where $\bar{G}(x) = 1 - G(x)$ is any tail distribution function and $m(u)$ is the renewal intensity. This convolution appears in application formulas such as for the availability in an alternating renewal process, the mean number present in a $GI/G/\infty$ queue, and the distribution of the local variables forward and backward recurrence time. Since the forward recurrence time has the widest appeal to applications generally, we will illustrate both proposed approximations in the context of an approximation to the forward recurrence time distribution.

The Key Renewal Theorem assures that, since the interevent distribution F is assumed to be non-lattice, the limiting value for the convolution in question is

$$\lim_{t \to \infty} \int_0^t \bar{G}(t-u)m(u)du = \mu \cdot \lambda \ ,$$

where

$$\mu = \int_0^\infty \bar{G}(x)dx \quad (<\infty) \ .$$

(Note: μ is the mean for distribution G .)

To obtain our first approximation to the convolution $c(t)$, we can simply replace $m(u)$ by $h_1(u)$ or $h_2(u)$, and use numerical integration to obtain values. Since h_1 is generally a closer approximation, we take

$$c(t) \cong \int_0^t \bar{G}(t-u)h_1(u)du \tag{23}$$

as an approximation to $c(t)$. As we seen in the numerical illustrations below, this can be a very close approximation when used for the forward recurrence time distribution.

Another approximation to $c(t)$ is easily obtained as follows. By considering G to be the distribution of X_1 in the renewal process, and M_1 to be the corresponding renewal function, we have the following version of equation (1b)

$$c(t) = \int_0^t \bar{G}(t-u)m(u)du = G(t) + M(t) - M_1(t) . \tag{24}$$

By replacing $M(t)$ by $A_2(t)$ (see equation (8)) and $M_1(t)$ by an analogous approximation for the general case, we have

$$M(t) \cong \lambda t + (1+k)F(t) - F_e(t)$$

and

$$M_1(t) \cong \lambda t - F_e(t) + G(t) + [1+k-\lambda\mu]F(t) ,$$

which upon substitution into equation (24) yield

$$c(t) \cong \lambda\mu F(t) \tag{25}$$

as an approximation to $c(t)$.

Applying these approximations to the distribution of forward recurrence time, Y_t , we have

$$\text{Prob}\{Y_t > y\} \cong \bar{F}(t+y) + \int_0^t \bar{F}(t+y-u)\{f(u) + \frac{F^2(u)}{\int_0^u \bar{F}(x)dx}\}du$$

as the first approximation. Values are obtained by using numerical integration.

To apply the second approximation to the forward recurrence time distribution, we can write

$$\text{Prob}\{Y_t > y\} = \bar{F}(t+y) + \bar{F}(y) \int_0^t \bar{G}(t-u)m(u)du$$

where

$$\bar{G}(x) = \bar{F}(x+y)/\bar{F}(y) ,$$

That is, G is the conditional distribution of residual life beyond y. In this case, μ is the mean residual life, or

$$\mu = \int_0^\infty \frac{\bar{F}(x+y)}{\bar{F}(y)} dx = \frac{\int_y^\infty \bar{F}(x)dx}{\bar{F}(y)} .$$

Now, substitution of equation (24) into the above equation yields

$$\text{Prob}\{Y_t > y\} \cong \bar{F}(t+y) + \bar{F}_e(y)F(t) . \tag{26b}$$

The following table illustrates the two approximations in a particular case.

y	Hyper-Exponential Interevent Times	2-Erlang Interevent Times
	λ = 1.0 c = 1.58	λ = 1.0 c = .707
	t = 0.5	t = 0.5
0.	1.0 1.0 1.0	1.0 1.0 1.0
0.2	0.765 0.765 0.804	0.823 0.827 0.804
0.4	0.600 0.600 0.663	0.653 0.657 0.629
0.6	0.483 0.483 0.558	0.506 0.509 0.482
0.8	0.398 0.398 0.479	0.385 0.388 0.363
1.0	0.335 0.335 0.419	0.289 0.291 0.271
1.5	0.236 0.236 0.314	0.135 0.135 0.125
2.0	0.179 0.179 0.247	0.060 0.060 0.055
2.9	0.119 0.119 0.167	0.013 0.013 0.012

TABLE 6. Forward Recurrence Time Distribution. Entries are Prob$\{Y_t > y\}$; 1st number is exact value, 2nd number using (26a), 3rd using (26b).

VII. CALCULATIONS IN DISCRETE RENEWAL PROCESSES

In many practical applications time is measured in discrete units, and the assumption of continuous distributions is an approximation. Renewal processes find application in inventory theory (for example see [Arrow, Karlin and Scarf, 1958] where the variable t is not "time" but discrete items of some commodity. In this section we present a simple computational method for solving renewal equations with discrete distributions and show some computational results.

Let X_i be the time between renewals $(i-1)$ and i, and let

$$F_n = \text{Prob}\{X_n = n\}, \quad n = 0, 1, 2, \ldots,$$

be the probability mass function, the same for all i. The ideas are first illustrated in obtaining the solution to the renewal function, equation (1). In the discrete case let M_n be the expected number of renewals up to an including time n, and $F_n = \sum_{j=0}^{n} = \text{Prob}\{X_n \le n\}$. Then equation (1) can be written

$$M_n = F_n + \sum_{j=0}^{n} M_{n-j} F_j, \quad n = 0, 1, 2, \ldots . \tag{1}$$

For any fixed n, define the $(n+1) \times (n+1)$ lower triangular matrix

$$F = \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ f_1 & f_0 & & \vdots \\ & & \ddots & 0 \\ f_n & & & f_0 \end{bmatrix}$$

and let F and M be the (n+1) column vectors

$$F = [F_0, F_1, F_2, \cdots, F_n] ,$$

and

$$M = [M_0, M_1, M_2, \cdots, M_n]$$

respectively. Then equation (1) can be re-written as

$$M = F + \mathcal{F}M .$$

Note that $\mathcal{F}$ is a non-negative matrix, every row sum is ≤ 1 , and at least one row sums to strictly less than 1 (assuming $E[X] > 0$) . Therefore $(I-\mathcal{F})^{-1}$ exists, where I is the identity matrix, and

$$M = (I-\mathcal{F})^{-1}F .$$

The inverse matrix of $(I-\mathcal{F})$ has the structure

$$G = \begin{bmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & \ddots & 0 \\ g_n & \cdots & \cdots & g_0 \end{bmatrix} ,$$

and so is completely determined by a vector $g = [g_0, \ldots, g_n]$.

For small n G is probably quite easily calculated using a general matrix inversion routine. The primitive operator $\boxed{\div}$ in the APL language is extremely efficient. But for large n it is advantageous to make use of the special structure of $\mathcal{F}$.

Let us assume that (n+1) is a power of 2. If not we increase it to the next higher power. For example if (n+1) is 100 increase it to $128 = 2^7$. Thus let us assume j is given by $2^j = (n+1)$, and write F_j for the $(n+1) \times (n+1)$ matrix F, G_j for the matrix G. If we partition F_j into

$$F_j = \begin{bmatrix} F_{j-1} & 0 \\ H_{j-1} & F_{j-1} \end{bmatrix}, \; j = 1, 2, \ldots$$

with $F_0 = F_0$ and $H_0 = F_1$, then

$$G_{j+1} = \begin{bmatrix} G_j & 0 \\ (G_j \; H_j \; G_j) G_j & \end{bmatrix} \qquad j = 0, 1, 2, \ldots ,$$

with $G_0 = (1/1-f_0)$.

Using an array manipulation language such as APL one can calculate large dimensioned matrices G in only a few iterations with no matrix inversion necessary.

Figure 1 shows the renewal function for X_i distribution binomially with parameters 100 and .2. Thus $E(X_i) = 20$, $c = .2$ and $k = -.48$. The approximation $A_1(t) = .05t-.48$ is shown for comparison. A four-instruction program written in APL took only 1.8 seconds on an IBM 370 model 158 to solve for M with $n = 200$ (10 mean times).

Having determined a method of solving (1) we have really solved a whole class of problems in renewal theory. In place of F let

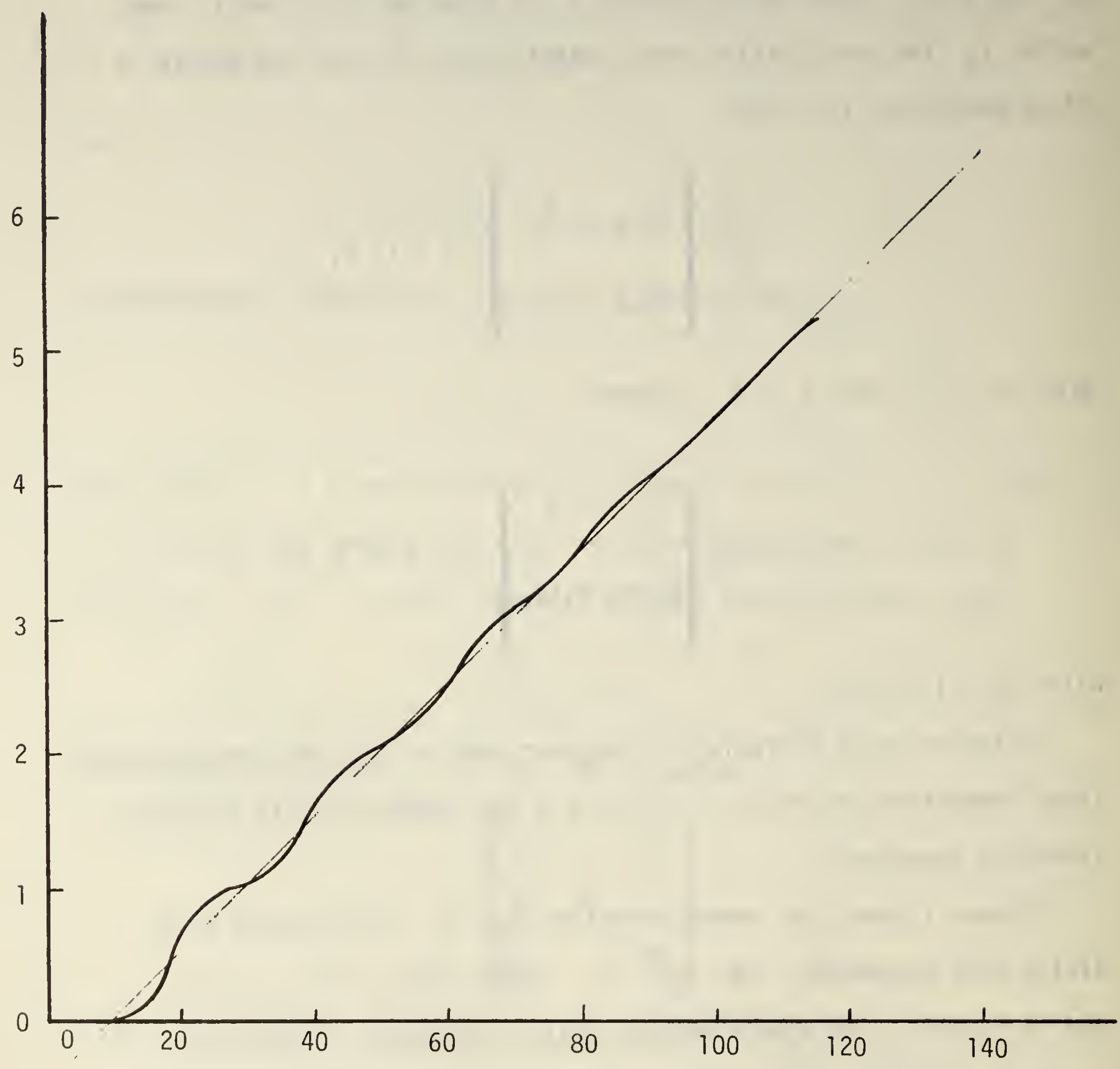

FIGURE 1: Renewal Function for a Binomial Process with Parameters 100 and 2.

us write a vector $h = [h_0, h_1, . . . , h_n]$. Now consider the renewal equation for the $(n+1)$ - vector r ,

$$r = h + Fr ,$$

where F is the same matrix as before. Then

$$r = Gh .$$

By carefully choosing h a number of problems can be solved. Let $h = [f_0, f_1, \ldots , f_n]$. Then r is a vector of renewal intensities. Figure 2 shows the renewal intensity for a binomial renewal process with parameters 100 and .2 . As a further example consider the distribution of the excess random variable. Let Y_n be the time until the first renewal after time n , and $\bar{G}(j,n) = \text{Prob}\{Y_n > j\}$, $j = 0, 1, 2, \ldots$. Define for a fixed j $\bar{G}_j = [\bar{G}(j, 0) , \bar{G}(j, 1) , \ldots , \bar{G}(j, n)]$. If we let $h = [\bar{F}_j, \bar{F}_{j+1}, \bar{F}_{j+2}, \ldots , \bar{F}_{j+n}]$, then r will give us $\bar{G}_j$. In this way we can determine for a fixed j how the excess distributions are converging with increasing n .

Consider an alternating renewal process as described in Section IV. Let A_n be the probability the system is operating at time n . Let the "up" times be distributed with distribution function $U_j = P[U_d \text{ time} \le j]$. Let F_j be the probability mass function of the sum of an up time and a down time, and F the same as above. If h is taken to be the vector $[\bar{U}_0, \bar{U}_1, \ldots , \bar{U}_n]$ then r will give the availabilities (assuming the system starts in the "up" state) at times $0, 1, \ldots , n$.

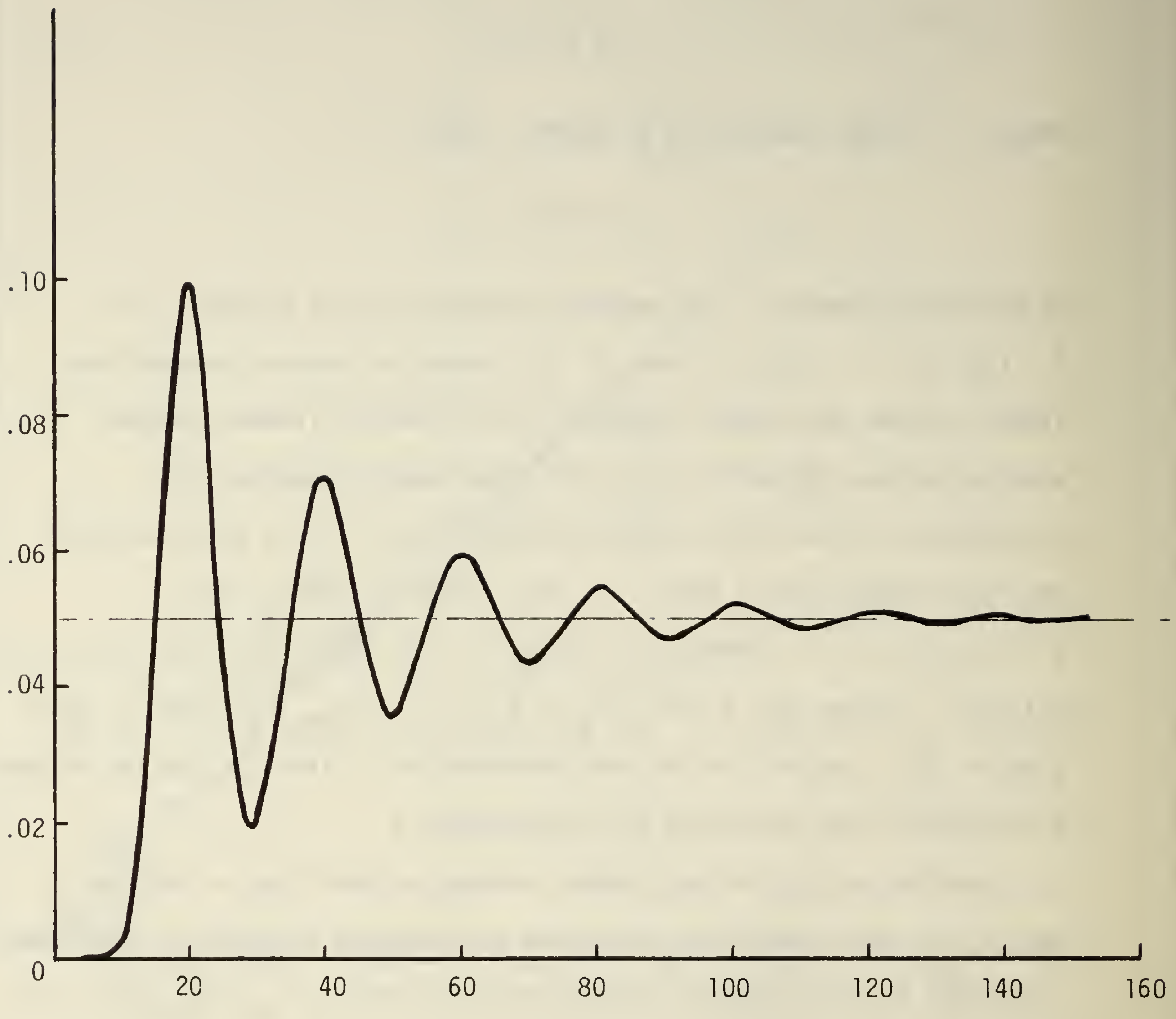

FIGURE 2: Renewal Intensity for a Binomial Process with Parameters 100 and .2 .

VIII. MISCELLANY.

In this section we mention a few topics of special interest. These topics are higher moments of the renewal counting variable, some generalizations of the renewal model and superposition of several renewal processes.

In some applications, higher moments of the renewal counting variable N_t may be of interest. In particular, the variance of N_t may be desired. Consider, for example, a situation in which events occur at times $t_i = i \cdot a + e_i$, where $e_1 e_2, \ldots$, are IID, normal $(0, \sigma)$, as i runs over all intergers. This may be the arrival pattern of persons with appointments. This is clearly not a renewal process (unless $\sigma = 0$), however, when σ is large, the point process appears, over fixed time periods, similar to a Poisson process. To distinguish this point process from the Poisson process or perhaps from other renewal processes, the variance of N_t can be investigated. The variance of N_t is also used in the analysis of the pooled output of several renewal processes, a topic mentioned below.

Higher moments of N_t are treated in [Cox, 1962, pp. 59-60], where the Laplace transform methods are used. The r^{th} semi-invariant moment of N_t is shown to be asymptotic to $\lambda_r t + V_r + O(1)$, (see also [Smith, 1959]) where λ_r is a function of the first r moments of the interevent distribution, and V_r is a function of the first $r + 1$ moments. In particular, the variance of N_t is $(\lambda\sigma)^2 \lambda t + o(t)$ as t tends to ∞.

Since the quantity $(\lambda\sigma)^2$ appears as a factor times the variance of a Poisson Process with the same rate, it is reasonable in practical investigations to use the coefficient of variation, $\lambda\sigma$, as a measure of variation in the counting processes N_t . Accordingly, when $\lambda\sigma < 1$ (> 1) , we say the process is "under variance" ("over variance"), the boundary or bench mark case being the Poisson process for which $\lambda\sigma = 1$.

Some bounds on higher moments are provided in [Barlow & Proschan, 1964], where it is proven that

$$\frac{E[N_t(N_t-1) \ldots (N_t-k+1)]}{k!} = E\binom{N_t}{k} \leq (\geq) \frac{(\lambda t)^k}{k!} ,$$

when we have that F is NBUE (UBNE) respectively. It is also shown that the variance can be bounded by the renewal function, that is,

$$V(N_t) \leq (\geq) E(N_t) = M(t)$$

when we have that F has increasing (decreasing) failure rate respectively.

One generalization of the renewal process model discussed by Smith [1958] and Cox [1962] is the cumulative process. In this model, a random amount W_i is contributed at each renewal epoch t_i . We are primarily concerned with the net contribution made by time t . If Z_t represents this net contribution, we have

$$Z_t = \sum_{i=1}^{N_t} W_i ,$$

where the empty sum is given value zero. If the W_i are identically equal to one, then $Z_t = N_t$. If W_i is the down time resulting from the i^{th} replacement of a failed part, Z_t is the down time, cumulative to t . The W_i may assume positive or negative values, if for example W_i

is a monetary return associated with the i^{th} event. The mean and variance of Z_t, when the W_i are iid, and independent of N_t, are given in [Cox, 1962, pp. 94] as

$$E(Z_t) = M(t) \cdot \mu_w$$

$$V(Z_t) = M(t) \cdot \sigma_m^2 + V(N_t)\mu_w^2$$

where μ_w and σ_w^2 are the mean and variance respectively of the W_i distribution. Asymptotic normality of Z_t is also shown, with the asymptotic expansions for $M(t)$ and $V(N_t)$ replacing the exact values in the above expressions.

In [Barlow & Proschan, 1964], the generalization in which the successive times between events may have different distributions, but must have a common mean, is discussed. Interevent times continue to be an independent set. The bound

$$M(t) < (>) \lambda t$$

is proven, when all interevent distributions have increasing (decreasing) failure rate respectively. These hypotheses can be relaxed to NBUE (UBNE) respectively.

The superposition of several renewal processes provides a point process distinct from a renewal process, (except in the Poisson Process case). Such processes are observed when two or more arrival streams are pooled, for example, or when failures of a system are the failures of any component, and each component fails according to some renewal process.

The asymptotic result for many independent contributing processes is that, over intervals of fixed length, the composite process exhibits the properties of a Poisson process. The contributing processes may be non-identical, but the relative contribution to the total rate made by each contributing process must tend uniformily to zero. See [Drenick, 1960] for a complete statement of the necessary conditions. The superposition of only a few processes is treated in [Cox, 1962, pp. 71-77], wnere distributional results are emphasized, and in [Cox and Lewis, 1966, pp. 210-223] where a statistical analysis is given as well. See also [Cox and Smith, 1954] for a complete discussion.

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